Announcements

1) Math Club Talk
this room, 4:00
David James on
the SVD

Gaussian Elimination

Something Familiar!

This is the classical method for solving systems of linear equations

Example 1: Solve

$$\begin{bmatrix} 2 & 1 & 5 \\ 6 & 12 & 0 \\ 3 & -4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

using Gaussian elimnation.

Form the augmented matrix

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 6 & 12 & 0 & 3 \\ 3 & -4 & 7 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 6 & 12 & 0 & 3 \\ 3 & -4 & 7 & -2 \end{bmatrix}$$

$-3R1 + R2, \; -\frac{3}{2}R1 + R3$

$$= \begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & 9 & -15 & 6 \\ 0 & -11/2 & -1/2 & -1/2 \end{bmatrix}$$

$\frac{11}{18} R_2 + R_3$

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & 9 & -15 & 6 \\ 0 & -11/2 & -1/2 & -1/2 \end{bmatrix}$$

$$\frac{11}{18}R_2 + R_3$$

upper triangular

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & 9 & -15 & 6 \\ 0 & 0 & -174/18 & 57/18 \end{bmatrix}$$

Solve for x, y, z using back substitution.

The Idea: this algorithm is primitive; at the first stage, if $A=(a_{i,j})_{i,j=1}^{n}$, you eliminate the entries below $a_{1,1}$ by adding multiples of the first row. At the second stage, you have $B=(b_{i,j})_{i,j=1}^{n}$, eliminate the entries below $b_{2,2}$ in the same manner.

Keep going to eventually acheive an upper-triangular matrix and a single additional column, which you then solve via back substitution.

Back to Example 1: (LU)

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 6 & 12 & 0 & 3 \\ 3 & -4 & 7 & -2 \end{bmatrix} = C$$

$-3R1+R2$, $-3/2\,R1+R3$

Write as a matrix acting on C.

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 6 & 12 & 0 & 3 \\ 3 & -4 & 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & 9 & -15 & 6 \\ 0 & -11/2 & -1/2 & -1/2 \end{bmatrix}$$

Next we added $\frac{11}{18}$ R_2 to R_3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{11}{18} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 5 & -1 \\ 6 & 12 & 0 & 3 \\ 3 & -4 & 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & 9 & -15 & 6 \\ 0 & 0 & -\frac{174}{18} & \frac{57}{18} \end{bmatrix}$$

If $M =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11/18 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 6 & 12 & 0 \\ 3 & -4 & 7 \end{bmatrix},$$

$$U = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 9 & -15 \\ 0 & 0 & -174/18 \end{bmatrix}$$

Then $\det(M) = 1$, so

since $MA = U$,

if we multiply on

the left by $M^{-1} = L$,

we get

$$\boxed{A = LU}$$

(the L-U decomposition)

Note: failure of algorithm

whenever the k^{th} diagonal entry at the k^{th} stage is zero.

Example 2:

Solve

$$\begin{bmatrix} 0 & 5 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

via Gaussian elimination:

divide by zero at the first stage!

Issues with Gaussian elimination

1) Dividing by zero

2) Algorithms for LU-decomposition are not, in general, backwards stable.

Don't use this!

Pivoting

Interchanging rows & columns to make stability attainable

Complete vs. Partial Pivoting

Complete: row & column interchanges

Partial: Only row interchanges

Example 3: (partial pivoting)

Solve

$$\begin{bmatrix} 0 & 5 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Augment:

$$\begin{bmatrix} 0 & 5 & 10 \\ 2 & 8 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 & 10 \\ 2 & 8 & 20 \end{bmatrix}$$

Interchange R1 with R2

$$\begin{bmatrix} 2 & 8 & 20 \\ 0 & 5 & 10 \end{bmatrix}$$

Solve using back substitution.

We applied the permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ on the

left.

In general we have an LU decomposition associated with partial pivoting,

$$PA = LU$$

where P is a permutation matrix.

Example 4: (partial vs. full pivoting)

$$\begin{bmatrix} 5 & -1 & 4 \\ 12 & 3 & 2 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ -9 \end{bmatrix}$$